Exercise 6

Use power series to solve the differential equation.

$$y'' = y$$

Solution

$x = 0$ is an ordinary point, so the ODE has a power series solution centered here.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Differentiate the series with respect to x.

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Differentiate the series with respect to x once more.

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substitute these formulas into the ODE.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} a_n x^n$$

Make the substitution $n = k + 2$ in the series on the left and the substitution $n = k$ in the series on the right.

$$\sum_{k+2=2}^{\infty} (k+2)[(k+2)-1] a_{k+2} x^{(k+2)-2} = \sum_{k=0}^{\infty} a_k x^k$$

Simplify the series on the left.

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k = \sum_{k=0}^{\infty} a_k x^k$$

For the series to be equal, the coefficients of x^k must be equal as well.

$$(k+2)(k+1) a_{k+2} = a_k$$

Solve for a_{k+2}.

$$a_{k+2} = \frac{1}{(k+2)(k+1)} a_k$$

In order to determine a_k, plug in values for k and try to find a pattern.

$$k=0: \quad a_2 = \frac{1}{(0+2)(0+1)}a_0 = \frac{1}{2\cdot 1}a_0$$

$$k=1: \quad a_3 = \frac{1}{(1+2)(1+1)}a_1 = \frac{1}{3\cdot 2}a_1$$

$$k=2: \quad a_4 = \frac{1}{(2+2)(2+1)}a_2 = \frac{1}{4\cdot 3}\left(\frac{1}{2\cdot 1}a_0\right) = \frac{1}{4\cdot 3\cdot 2\cdot 1}a_0$$

$$k=3: \quad a_5 = \frac{1}{(3+2)(3+1)}a_3 = \frac{1}{5\cdot 4}\left(\frac{1}{3\cdot 2}a_1\right) = \frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1}a_1$$

$$\vdots$$

The general formula for the even subscripts is

$$a_{2m} = \frac{a_0}{(2m)!},$$

and the general formula for the odd subscripts is

$$a_{2m+1} = \frac{a_1}{(2m+1)!}.$$

Therefore, the general solution is

$$\begin{aligned} y(x) &= \sum_{m=0}^{\infty} a_m x^m \\ &= \sum_{m=0}^{\infty} a_{2m}x^{2m} + \sum_{m=0}^{\infty} a_{2m+1}x^{2m+1} \\ &= \sum_{m=0}^{\infty} \frac{a_0}{(2m)!}x^{2m} + \sum_{m=0}^{\infty} \frac{a_1}{(2m+1)!}x^{2m+1} \\ &= a_0 \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} \frac{x^{2m+1}}{(2m+1)!} \\ &= a_0 \cosh x + a_1 \sinh x, \end{aligned}$$

where a_0 and a_1 are arbitrary constants.